On the Topological Complexity of DC-Sets

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Abstract. A DC-set is a set defined by means of convex constraints and one additional inverse convex constraint. Given an arbitrary closed subset M of the Euclidean n-space, we show that there exists a DC-set in $(n+1)$-space being homeomorphic to the given set M. Secondly, for any fixed $n \geqslant 2$, we construct a compact n-dimensional manifold with boundary, which is a DC-set and which has arbitrarily large Betti-numbers r_k for $k \leqslant n-2$.

Key words. Reverse convex constraint, DC-set, topological complexity.

1. Introduction and Main Results

In optimization it happens that one has to optimize a real valued function over a set which is described by means of convex constraints and a reverse convex constraint (see, for example [6]). The aim of this paper is to show how complex the topological structure of such sets might be.

DEFINITION 1. A subset A of $\mathbb{R}^n$ is called a DC-set ("difference of convex sets") if there exist convex functions $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$A = \{x \in \mathbb{R}^n | f(x) \leqslant 0,\ g(x) \geqslant 0\} \qquad \Box \qquad (1)$$

REMARK 1. In (1) we may replace the inequality $f(x) \leqslant 0$ by means of a finite number of them. This is due to the well-known fact that the maximum function taken over a finite number of convex functions, is convex too. Moreover, every DC-set can be represented as a lower level set of a DC-function ("difference of convex functions"). In fact, we have $A = \{x \in \mathbb{R}^n | h(x) \leqslant 0\}$, where h is the difference of the convex function $\max(f, 0) + \max(g, 0)$ and the convex function g.

THEOREM A. *Let $M \subset \mathbb{R}^n$ be a closed set. Then, there exists a DC-set $A \subset \mathbb{R}^{n+1}$ which is homeomorphic with M.* $\Box$

In the second theorem we use singular homology over the field of real numbers (for details we refer to [3], [5]). Let X be a topological space. The dimension r_k of

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the k-th homology space of X is called the k-th Betti-number. The number r_0 is equal to the number of (path-) connected components of X. The number r_k, $k \geqslant 1$, counts, roughly speaking, the number of $(k+1)$-dimensional "holes" in X. In case that X is a compact n-dimensional manifold (with boundary), then all Betti-numbers are finite, and, in particular, $r_k = 0$ for $k > n$.

THEOREM B. *Let $n \geqslant 2$ be a fixed integer, and let $p_0, p_1, \ldots, p_{n-2}$, be arbitrary non-negative integers. Then, there exists a compact DC-set $A \subset \mathbb{R}^n$, being an n-dimensional manifold (with boundary), such that for its Betti-numbers the inequalities $r_k \geqslant p_k$, $k = 0, 1, \ldots, n-2$, hold.* □

2. Proof of Theorems A, B

LEMMA 1 (cf. [1]). *Let $M \subset \mathbb{R}^n$ be a closed set. Then, there exists a non-negative C^∞-function $h: \mathbb{R}^n \to \mathbb{R}$ with $M = h^{-1}(0)$.* □

LEMMA 2 (cf. [4]). *Let $h: \mathbb{R}^n \to \mathbb{R}$, be a C^2-function. Then, there exist convex C^2-functions f, $g: \mathbb{R}^n \to \mathbb{R}$ such that $h = f - g$.* □

Proof of Theorem A. Let $M \subset \mathbb{R}^n$ be a closed set. In virtue of Lemma 1 we choose a non-negative C^∞-function $h: \mathbb{R}^n \to \mathbb{R}$ with $M = h^{-1}(0)$. By Lemma 2, the function h can be written as the difference $h = f - g$ of convex C^2-functions f, g. With the aid of an additional coordinate x_{n+1}, we define the following two convex functions F, G:

$$F(x, x_{n+1}) = f(x) - x_{n+1},\ G(x, x_{n+1}) = g(x) - x_{n+1}, \text{ where } x = (x_1, \ldots, x_n).$$

Consider the DC-set $A = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} | F(x, x_{n+1}) \leqslant 0,\ G(x, x_{n+1}) \geqslant 0\}$. Note, that $f(x) \geqslant g(x)$ for all x, and that equality precisely holds on the set M. Consequently, the set A exactly consists of those points (x, x_{n+1}) of Graph(F) with $x \in M$, where Graph(F) $:= \{(x, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} = F(x)\}$ denotes the graph of F. Hence, $M = \Pi(A)$, where $\Pi: (x, x_{n+1}) \mapsto x$ denotes the natural projection. The proof is now established, since the restricted mapping $\Pi|_{\text{Graph}(F)}: \text{Graph}(F) \to \mathbb{R}^n$ is a homeomorphism. □

Before proving Theorem B we need to introduce some concepts of Karush–Kuhn–Tucker theory, and also we need a theorem on homeomorphism and alternatives, based on Morse theory. For details we refer to [2] and [3].

Let $h_i, g_j: \mathbb{R}^n \to \mathbb{R}$, $i \in I$, $j \in J$, be a finite number of C^1-functions, and define the set $M[h, g]$:

$$M[h, g] = \{x \in \mathbb{R}^n | h_i(x) = 0, i \in I, g_j(x) \geqslant 0, j \in J\}. \tag{2}$$

Let $J_0(x) = \{j \in J | g_j(x) = 0\}$ denote the set of active inequality constraints. We say that the linear independence constraint qualification (LICQ) is satisfied at $\bar{x} \in M[h, g]$ if the set of vectors $\{Dh_i(\bar{x}),\ i \in I,\ Dg_j(\bar{x}),\ j \in J_0(\bar{x})\}$ is linearly independent. Here, $Dh_i(\bar{x})$ stands for the row vector of the first order partial derivatives of the function h_i, evaluated at point $\bar{x}$.

The Mangasarian–Fromovitz constraint qualification (MFCQ) is said to hold at $\bar{x}$ if the set of vectors $\{Dh_i(\bar{x}),\ i \in I\}$ is linearly independent, and if, in addition, there exists a vector $\xi \in \mathbb{R}^n$ with $Dh_i(\bar{x})\xi = 0,\ i \in I$, and $Dg_j(\bar{x})\xi > 0,\ j \in J_0(\bar{x})$. We say that (MFCQ) holds on $M[h, g]$ if (MFCQ) is satisfied at each point $\bar{x} \in M[h, g]$. If it is clear, which constraint functions h_i, g_j are meant, we just say: (MFCQ) holds on M; here $M = M[h, g]$. In case that (MFCQ) holds on $M[h, g]$, then $M[h, g]$ is an $(n - |I|)$-dimensional manifold (with boundary).

ASSUMPTION. From now on we assume that (MFCQ) holds on $M[h, g]$, and that the functions $f, h_i, g_j: \mathbb{R},\ i \in I,\ j \in J$, are C^2-functions.

A point $\bar{x} \in M[h, g]$ is called a Karush–Kuhn–Tucker point (KKT-point) for $f|_{M[h,g]}$, if there exist real numbers $\lambda_i, i \in I$, and $\mu_j,\ j \in J_0(\bar{x})$, such that

$$Df = \sum_{i \in I} \lambda_i Dh_i + \sum_{j \in J_0(\bar{x})} \mu_j Dg_j|_{\bar{x}} \,. \tag{3}$$

Note, that the above numbers λ_i, μ_j are unique in the case that (LICQ) holds. A KKT-point $\bar{x}$ is called non-degenerated if (LICQ) holds at $\bar{x}$, and if, in addition, $\mu_j > 0,\ j \in J_0(\bar{x})$, and the matrix $V^T D^2 L(\bar{x})\ V$ is non-singular. In the latter, $D^2 L(\bar{x})$ stands for the Hessian of the Lagrange function L at $\bar{x}$,

$$L(x) = f(x) - \sum_{i \in I} \lambda_i h_i(x) - \sum_{j \in J_0(x)} \mu_j g_j(x) \,, \tag{4}$$

and V is a matrix whose columns form a basis for the tangent space $T(\bar{x})$,

$$T(\bar{x}) = \bigcap_{i \in I} Ker\, Dh_i(\bar{x}) \cap \bigcap_{j \in J_0(\bar{x})} Ker\, Dg_j(\bar{x}) \,. \tag{5}$$

Note, that $D^2 L = D^2 f$ in the case of affine linear constraint function h_i, g_j. This will tacitly be used in the proof of Theorem B.

If $\bar{x}$ is a non-degenerated KKT-point for $f|_{M[h,g]}$, then the number of negative eigenvalues of the above matrix $V^T D^2 L(\bar{x})\ V$ is called the (quadratic) index at $\bar{x}$.

With $M = M[h, g]$ and $b \geq a$, we put

$$M^b = \{x \in M | f(x) \leq b\} \,, \quad M_a^b = \{x \in M | a \leq f(x) \leq b\} \,. \tag{6}$$

Regarding the next lemma, ref. [3] and ref. [2] are based on (LICQ) and (MFCQ), respectively.

LEMMA 3 (cf. [2], [3]). *Under the above notations and assumption, suppose that M is compact. In case that M_a^b, with $b \geqslant a$, contains no KKT-points, then the lower level sets M^a and M^b are homeomorphic. On the other hand, suppose that M_a^b contains precisely one KKT-point $\bar{x}$ and we have $a < f(\bar{x}) < b$. If, in addition, the KKT-point $\bar{x}$ is non-degenerated with quadratic index k, then the following alternative holds (in case $k=0$, always the second one holds)*:

$$\text{either} \begin{Bmatrix} r_{k-1}(M^b) = r_{k-1}(M^a) - 1 \\ r_k(M^b) = r_k(M^a) \end{Bmatrix} \text{ or } \begin{Bmatrix} r_{k-1}(M^b) = r_{k-1}(M^a) \\ r_k(M^b) = r_k(M^a) + 1 \end{Bmatrix}.$$

Moreover, $r_i(M^b) = r_i(M^a)$ for $i \notin \{k-1, k\}$. □

Proof of Theorem B. The required DC-set A is obtained as follows. Firstly, we construct a special polytope P^*. Then, the set A is defined as the intersection of P^* and a suitable upper level set of the strictly concave function $\phi := -\|x\|^2$, where $\|\cdot\|$ denotes the Euclidean norm.

Put $p = p_0 + p_1 + \cdots + p_{n-2}$ and let $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^n$. Choose p distinct points $x_{0,i}$, $i = 1, \ldots, p$, on the sphere S^{n-1}, and then choose $r < 1/4 \min\{1, \min_{i,j;i \neq j} \|x_{0,i} - x_{0,j}\|\}$. For $i = 1, \ldots, p$, we choose n points $v_{1,i}, \ldots, v_{n,i}$ in S^{n-1}, such that $\|x_{0,i} - v_{k,i}\| = r$, $k = 1, \ldots, n$, and such that the simplex $\Sigma(v_{1,i}, \ldots, v_{n,i})$ spanned by the set $\{v_{1,i}, \ldots, v_{n,i}\}$ is a regular one. Now, we shift the tangent space $T_{v_{k,i}}$ of the sphere S^{n-1} at the point $v_{k,i}$ parallelly until it meets the point $x_{0,i}$, $k = 1, \ldots, n$. This uniquely defines a polyhedral cone C_i, pointed at $x_{0,i}$ and containing the origin. The intersection of C_i with the hyperplane through the points $v_{k,i}$, $k = 1, \ldots, n$, is again a regular simplex with vertices denoted by $x_{k,i}$, $i = 1, \ldots, n$. Note, that $\|x_{k,i}\| < 1$ for $k \neq 0$. Put $P = \mathscr{C}\{x_{k,i} \mid k = 0, 1, \ldots, n;\ i = 1, \ldots, p\}$, the convex hull of the points $x_{k,i}$. For the description of P we choose a minimal system of linear inequalities $a_j^{\mathsf{T}} x - b_j \geqslant 0$, $j \in J$. With respect to the latter system (MFCQ) holds. From the choice of the above number r, it follows that each segment $[x_{0,i}, x_{k,j}]$, $i \neq j$, is contained in the cone C_i. In view of the very construction, this implies that even (LICQ) is fulfilled at $x_{0,i}$, $i = 1, \ldots, p$.

Let Vert(P) denote the set of vertices of P.

Put $\alpha = \max\{\|x\| \mid x \in \text{Vert}(P) \backslash \{x_{0,1}, \ldots, x_{0,p}\}\}$, $\beta = \max\{\|x\| \mid x \text{ is KKT–point for } \phi|_P \text{ and } \|x\| \neq 1\}$, and choose γ such that $\max\{\alpha, \beta\} < \gamma < 1$. For each C_i we choose a hyperplane H_i orthogonal to the vector $x_{0,i}$, intersecting the open line segment $(0, x_{0,i})$ and satisfying the inequality $\inf\{\|x\| \mid x \in H_i\} > \gamma$.

This hyperplane intersects the extremal rays of C_i, say at the points $y_{1,i}, \ldots, y_{n,i}$. Note, that the simplex $\Sigma(y_{1,i}, \ldots, y_{n,i})$ is regular. For appropriate choice of the hyperplanes H_i, the simplices $\Sigma(x_{0,i}, y_{1,i}, \ldots, y_{n,i})$, $i = 1, \ldots, p$, are pairwise disjoint.

Now, we consider the polytope $\tilde{P}$ defined by

$$\bar{P} = \mathscr{C}\{(\mathrm{Vert}(P)\setminus\{x_{0,i}|i=1,\ldots,p\}) \cup \{y_{k,i}|k=1,\ldots,n; i=1,\ldots,p\}\}.$$

Elementary calculations show that the barycenters of the k-faces of the simplex $\Sigma(y_{1,i},\ldots,y_{n,i})$, $k=0,1,\ldots,n-1$, are precisely the KKT-points for $\phi|_{\bar{P}}$ belonging to the simplex $\Sigma(y_{1,i},\ldots,y_{n,i})$; moreover, all of them are non-degenerated with quadratic index equal to the dimension of the corresponding face.

By means of local perturbation of the polytope P within the simplex $\Sigma(x_{0,i}, y_{1,i},\ldots,y_{n,i})$ around the point $x_{0,i}, i=p_0+1,\ldots,p$, we obtain the announced special polytope P^*. For suitable $\varepsilon>0$ the lower level set $\{x\in P^*|\phi(x)\leqslant\varepsilon-1\}$ then is the DC-set A we are looking for.

Let $1\leqslant k\leqslant n-2$, and suppose that the local perturbations are already performed around $(p_1+\cdots+p_{k-1})$ of the vertices $x_{0,i}, i=p_0+1,\ldots \Sigma_{j=1}^{k-1} p_j$. Choose a set of p_k unperturbed vertices of the type $x_{0,i}$, and let x_0 be one of these. We proceed by working within the simplex $\Sigma(x_0, y_1,\ldots,y_n)$. Put $\bar{y}_j=y_j$ for $j=k+3, k+4,\ldots,n$. On the open line segment (x_0, y_i) we choose a point $\bar{y}_i$, $i=1,\ldots,k+2$, in such a way that the affine hull of $\{\bar{y}_i|i=1,\ldots,k+2\}$ is parallel to the affine hull of $\{y_i|i=1,\ldots,k+2\}$. In this way, the points $\bar{y}_i$, $i=1,\ldots,k+2$, can be chosen arbitrarily close to the point x_0. The points $\bar{y}_1,\ldots,\bar{y}_n$ uniquely define a minimal closed halfspace containing these points but not x_0. The local perturbation around x_0, now, consists of taking the intersection of the latter halfspace with the polytope, perturbed so far. So, after this intersection is performed, we have arrived at a polytope, say $\bar{P}$. Let $\bar{F}_{k+1}$ be its $(k+1)$-dimensional face spanned by the vertices $\bar{y}_1,\ldots,\bar{y}_{k+2}$. The function ϕ takes its maximum over $\bar{F}_{k+1}$ in a point $\bar{x}$ in its relative interior, and we may suppose that the value $\phi(\bar{x})$ is close to $\phi(x_0)$. The point $\bar{x}$ is, in fact, a non-degenerated KKT-point for $\phi|_{\bar{P}}$ with quadratic index equal to $k+1$.

A moment of reflection shows that we can choose $\eta>0$ such that no other KKT-point for $\phi|_{\bar{P}}$ within the simplex $\Sigma(x_0, y_1,\ldots,y_n)$ has a ϕ-value in the interval $[\phi(\bar{x})-\eta, \ \phi(\bar{x})+\eta]$. Consider the polytope $K=\mathscr{C}\{y_1,\ldots,y_n, \bar{y}_1,\ldots,\bar{y}_{k+2}\}$. By K^a we denote the lower level set $\{x\in K|\phi(x)\leqslant a\}$. Put $C=K^{\phi(\bar{x})-\eta}$. Then, C is a compact manifold with boundary. We contend that its Betti-numbers $r_i(C)$ are given by $r_0(C)=r_k(C)=1$, and $r_i(C)=0$ for $i\not\in\{0,k\}$.

To see this, firstly note that the set $\{x\in K|\phi(x)\geqslant\phi(\bar{x})+\eta\}$ does not contain any KKT-point for $\phi|_K$. Let $\phi_{\max}$ denote $\max\{\phi(x)|x\in K\}$.

In virtue of Lemma 3 we then conclude that $K^{\phi_{\max}}$ and $K^{\phi(\bar{x})+\eta}$ are homeomorphic. But, $K^{\phi_{\max}}=K$, and K, being a compact convex set, is contractible. Hence, both K and $K^{\phi(\bar{x})+\eta}$ have the homology of a point, i.e. $r_0(K)=r_0(K^{\phi(\bar{x})+\eta})=1$ and $r_i(K)=r_i(K^{\phi(\bar{x})+\eta})=0$ for $i>0$. Since the KKT-point $\bar{x}$ has quadratic index equal to $k+1$, we must have (cf. Lemma 3) either $r_k(K^{\phi(\bar{x})+\eta})=r_k(C)-1$, or $r_{k+1}(K^{\phi(\bar{x})+\eta})=r_{k+1}(C)+1$. From the values of the Betti-numbers of $K^{\phi(\bar{x})+\eta}$ we conclude that the first alternative holds, thus $r_k(C)=1$. The other Betti-numbers of C are equal to the corresponding ones of $K^{\phi(\bar{x})+\eta}$. This proves our contention on the Betti-numbers of C.

Analogously, we can perform local perturbations around other $(p_k + p_{k+1} \cdots + p_{n-2} - 1)$ vertices of type $x_{0,i}$, still being unperturbed. During this procedure we adjust all values $\phi(\bar{x})$, with $\bar{x}$ as above, to one common value. The construction of the polytope P^* has now been completed.

Comparing P^* with P, we note that they have at least p_0 vertices on S^{n-1} in common. These vertices are global minima for the function ϕ on P^*. In fact, they produce in the lower level set $\{x \in P^* | \phi(x) \leq \varepsilon - 1\}$ enough (contractible) components in order to raise the Betti-number r_0. This completes the roof of the theorem. □

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